Tutorial 10. 16. Nov. 2015.

1. Let f be a holomorphic function on the disc D_{R_0} centered at the origin and of radius R_0.

(a) Prove that whenever $0 < R < R_0$ and $|z| < R$, then

$$f(z) = \frac{1}{2\pi}\int_0^{2\pi} f(Re^{i\varphi})\, \mathrm{Re}\left(\frac{Re^{i\varphi}+z}{Re^{i\varphi}-z}\right) d\varphi.$$

(b). Show that

$$\mathrm{Re}\left(\frac{Re^{i\gamma}+r}{Re^{i\gamma}-r}\right) = \frac{R^2-r^2}{R^2-2Rr\cos\gamma+r^2}$$

Hint:

$$0 = \int_{C_R} \frac{f(w)}{w-\frac{R^2}{\bar{z}}}\, dw$$

2. Show that $$\int_{-\infty}^{+\infty} \frac{dx}{(1+x^2)^{n+1}} = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2\cdot 4\cdot 6\cdots(2n)}\cdot\pi.$$

3. Suppose u is not an integer. Prove that

$$\sum_{n=-\infty}^{\infty} \frac{1}{(u+n)^2} = \frac{\pi^2}{(\sin \pi u)^2}.$$

by integrating

$$f(z) = \frac{\pi \cot \pi z}{(u+z)^2}$$

over the circle $|z| = R_N = N + 1/2$ (N integer, $N \geq |u|$), adding the residues of f inside the circle, and letting N tend to infinity.